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Abstract

Special functions play an important role in many diverse fields such as in Mathematical Analysis, Physics, Engineering and Allied Sciences. All the elementary functions such as logarithmic, exponential, trigonometric, algebraic and transcendental came under the category of special functions. The study of special functions grew up with the calculus and is consequently one of the oldest branches of analysis. It flourished in the nineteenth century as part of the theory of complex variables. In the second half of the twentieth century it has received a new impetus from a connection with Lie groups and a connection with averages of elementary functions.

In the present work, an attempt has been made to study the **SOLUTION OF SOME REMARKABLE PROBLEMS THROUGH HYPERGEOMETRIC APPROACH**. This thesis is spread over in **Ten Chapters**.

The aim of the **First Chapter** is to introduce several classes of special functions, which occur rather more frequently in the study of summations and transformations needed for the presentation of subsequent chapters. In this chapter, we have discussed Gamma function, Beta function, Pochhammer symbol and related results; Generalized hypergeometric function of one variable and associated results; Meijer's G-function and associated results; Appell's functions; General double hypergeometric functions of Kampé de Fériet and its convergence conditions; General double hypergeometric functions of Srivastava-Daoust and its convergence conditions; Some hypergeometric transformations and reduction formulas; Some hypergeometric summation theorem's; Some series identities; Some hypergeometric polynomials; Leibnitz's (also Leibniz) generalized rule for successive integration by parts; Leibnitz theorem for successive differentiation of the product of two functions; Gauss-Legendre's six-point formula; Some definite integrals; Curved surface area of a three-dimensional figure and many other functions.

In **Second Chapter**, we establish two new general double-series identities, which involve some suitably bounded sequences of complex numbers, by using Gessel-Stanton hypergeometric summation theorems for terminating Clausen series ${}_3F_2\left(\frac{3}{4}\right)$, ${}_3F_2\left(\frac{4}{3}\right)$ and series rearrangement technique. By using these two general double-series identities, we have derived five reduction formulas (numerically verified by using Mathematica software) for Srivastava-Daoust double hypergeometric functions with the

arguments $\left(z, \frac{-3z}{4}\right)$, $\left(z, \frac{-4z}{3}\right)$ $\left(\frac{-3z}{(1-z)}, \frac{-3z^2}{(1-z)^2}\right)$, $\left(\frac{-3z}{4(1-z)^2}, \frac{-z}{(1-z)}\right)$ and $\left(\frac{-3z}{(1-z)^2}, \frac{3z^2}{(1-z)^2}\right)$ in terms of generalized hypergeometric functions of one variable, under appropriate convergence conditions.

In **Third Chapter**, we obtain the closed forms of some reduction formulas (believed to be new) for Gauss functions ${}_2F_1[\alpha,\ \alpha+\frac{1}{2};\ 2\alpha-1;\ z], {}_2F_1[\alpha-1,\ \alpha-\frac{3}{2};\ 2\alpha-1;\ z], {}_2F_1[\alpha,\ \alpha-\frac{1}{2};\ 2\alpha+1;\ z],\ {}_2F_1[\alpha+1,\ \alpha+\frac{3}{2};\ 2\alpha+1;\ z],\ {}_2F_1[a,\ a\pm\frac{1}{2};\ 2a;\ \frac{4y^3}{(1-3y)^2}], {}_2F_1[a,\ a\pm\frac{1}{2};\ 2a;\ \frac{-4y^3}{(1-y)^2(1-4y)}]$ and Clausen functions ${}_3F_2[\gamma+1,\ \beta,\ \beta+\frac{1}{2};\ \gamma,\ 2\beta;\ z], {}_3F_2[\gamma+1,\ \beta,\ \beta-\frac{1}{2};\ \gamma,\ 2\beta;\ z]$,${}_3F_2[a,\ 3a-1,\ 3a-\frac{3}{2};\ a-1,\ 6a-2;\ y]$, ${}_3F_2[2a,\ 3a-1,\ 3a-\frac{3}{2};\ 2a-1,\ 6a-2;\ y]$ using series rearrangement technique. A result recorded in the table of Prudnikov et al.[Vol.3, p.460, Entry (103)] and a reduction formula of Joshi and Vyas [Published in International Journal of Mathematics and Mathematical Sciences 2005 (12), p.1921, Eq.(6.19)] is also modified here. As application of our closed forms of two reduction formulas (3.2.5) and (3.2.6), we establish four results for the reducibility of general Srivastava-Daoust double hypergeometric function in terms of generalized hypergeometric functions of one variable, with suitable convergence conditions.

In **Fourth Chapter**, we obtain a new general double-series identity involving the bounded sequence of arbitrary complex numbers using two-balanced Clausen summation theorem. Then among numerous application of this identity, we present a reduction formula for Srivastava-Daoust double hypergeometric function and a Clausen hypergeometric function ${}_3F_2$ having argument $\frac{-27z}{4(1-z)^3}$ with suitable convergence conditions. In this chapter, we also establish some new hypergeometric summation theorems (4.7.1), (4.8.1) and the results (4.9.1), (4.9.8), (4.9.15) and (4.9.22) (not recorded earlier and numerically verified using Mathematica software) for k-balanced terminating Clausen series in terms of the ratio of the product of Pochhammer symbols, by using a relation between two terminating Clausen series, decomposition of the ratio of two Pochhammer symbols, Chu-Vandermonde summation theorem, algebraic properties of gamma functions and Pochhammer symbols, representation of a linear, quadratic, cubic, bi-quadratic polynomials in terms of Pochhammer symbols and series rearrangement technique.

In **Fifth Chapter**, we obtain three hypergeometric generating relations associated with Kampé de Fériet double hypergeometric functions, by means of Gauss' quadratic transformation,Whipple's quadratic transformation, Kümmer's first transformation and Series rearrangement technique. Some special cases are also discussed. In this chapter, we also obtained a general double series identity having the bounded sequence, by using George Andrews summation theorem ${}_3F_2\left(\frac{3}{4}\right)$ and series rearrangement technique. Further, we specified a reduction formula for Srivastava-Daoust double hypergeometric function in terms of generalized hypergeometric function with argument $(27)^{(D-1-E)}z^3$.

In **Sixth Chapter**, we establish two identities which transforms one general double infinite series into a single series and another into a sum of three infinite single series, involving the bounded sequence of arbitrary complex numbers using Saalschütz summation theorem for terminating Clausen series. As application of our general double series identities, we present four cubic reduction formulas for Srivastava-Daoust double hypergeometric functions and a cubic transformation formula for the Clausen hypergeometric function ${}_3F_2$ with suitable convergence conditions. Some interesting and new special cases are also discussed. By the theory of analytic continuation, our cubic reduction formula (6.4.2) for special Srivastava-Daoust double hypergeometric function, is also valid in $\frac{-211}{25} \leq \Re(z) \leq \frac{3}{4}$ when $\Im(z) = 0$, using Mathematica software.

In **Seventh Chapter**, we obtain a general double-series identity involving the bounded sequence of arbitrary complex numbers. As application of our double series identity, we establish some reduction formulas for Srivastava-Daoust double hypergeometric function and Gaussian generalized hypergeometric function ${}_4F_3$. As special cases of our reduction formula for ${}_4F_3$ lead to some corollaries involving Clausen hypergeometric functions ${}_3F_2$. Making suitable adjustment of parameters in reduction formulas for ${}_4F_3$ and ${}_3F_2$, we obtain some results in terms of elementary functions and some special functions like Lerch generalized zeta function and incomplete Beta function. In this chapter, we also derive some summation formulas for three terminating series ${}_4F_3[-m,\ \alpha,\ \beta,\ 1+\gamma;\ 1+\alpha+m,\ 1+\alpha-\beta,\ \gamma;\ 1]$, ${}_4F_3[-m,\ \alpha,\ \beta,\ 1+\frac{\alpha}{2};\ \frac{\alpha}{2},\ 1+\alpha+m,\ 1+\alpha-\beta;\ 1]$ and ${}_4F_3[-m,\ \alpha,\ \beta,\ 1+\frac{\beta}{2};\ \frac{\beta}{2},\ 1+\alpha+m,\ 1+\alpha-\beta;\ 1]$. By using these summation formulas, we establish a reduction formula for Srivastava-Daoust double hypergeometric function having arguments $(\pm z)$ and some reduction formulas for Gaussian hypergeometric functions ${}_3F_2[\alpha,\ \beta,\ 1+\gamma; 1+\alpha-\beta, \gamma;\ \frac{-z}{(1+\sqrt{(1-z)})^2}]$, ${}_4F_3[\frac{\alpha}{2},\ \alpha,\ \beta,\ 1+\gamma;\ \frac{2+\alpha}{2},\ 1+\alpha-\beta,\ \gamma;\ \frac{-z}{(1+\sqrt{(1-z)})^2}]$ and ${}_5F_4[2\alpha,\ \frac{\alpha}{2},\ \alpha,\ \beta,\ 1+\gamma;\ 2\alpha-1,\ \frac{2+\alpha}{2},\ 1+\alpha-\beta,\ \gamma;\ \frac{-z}{(1+\sqrt{(1-z)})^2}]$, using series rearrangement techniques. A general double-series identity involving bounded sequence of essentially arbitrary complex numbers, is also given.

In **Eighth Chapter**, we provide the analytical solutions (not available in the literature) of some problems related with definite integrals involving the product of the derivatives of Legendre's polynomials of first kind having different order. As application of these integrals, we also obtain some summation theorems of Kampé de Fériet's double hypergeometric polynomials with arguments unity.

In **Ninth and Tenth Chapters**, we obtain the analytical expressions (not previously found and not recorded in the literature) for the exact curved surface area of a right elliptic single cone and elliptic paraboloid in terms of Appell's double hypergeometric function of second kind. The derivation is based on Mellin-Barnes type contour integral representations of generalized hypergeometric function ${}_pF_q(z)$, Meijer's G-function and analytic continuation formula for Gauss function. Moreover, we also obtain the analytical expression for the volume of the right elliptic single cone and elliptic paraboloid. Some special cases related to right circular cone and right circular paraboloid are also discussed. The closed forms for the exact curved surface area and volume of right elliptic single cone and elliptic paraboloid are also verified numerically by using *Mathematica Program.*